Indian Statistical Institute, Bangalore

M. Math.

First Year, Second Semester

Functional Analysis

Final Examination — Date: 24 April 2024

Maximum marks: 100 — Time: 10:00 AM–1:00 PM (3 hours)

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1. (a) Let $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ be Banach spaces. Suppose that
$$\exists \alpha \geq 0 : \ \|x\|_2 \leq \alpha \|x\|_1, \ \forall x \in X.$$
Show that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

   (b) Consider $X = C[0,1]$ in the L^p-norm $\|\cdot\|_p$ and sup norm $\|\cdot\|_\infty$. Show that if $1 \leq p < \infty$, then
$$\|f\|_p \leq \|f\|_\infty, \quad \forall f \in X,$$
but there is no $\beta > 0$ such that
$$\|f\|_\infty \leq \beta \|f\|_p, \quad \forall f \in X.$$
Does this contradict (a)? Justify your answer.

[5+15]

2. Find the dual of l^p for $1 < p < \infty$. Justify your answer. [15]

3. (a) Let H be a Hilbert space and $T : H \to H$ a linear operator which is symmetric, i.e.,
$$\langle Tx, y \rangle = \langle x, Ty \rangle, \quad \forall x, y \in H.$$
Prove that T is continuous.

OR

   (b) Let X, Y and Z be Banach spaces. Suppose that $T : X \to Y$ is linear, that $S : Y \to Z$ is linear, bounded and injective, and that $ST : X \to Z$ is bounded. Show that T is also bounded.

[13]

4. Let X be a Banach space and $\mathcal{K}(X)$ denotes the set of all compact operators on X. If $T \in \mathcal{K}(X)$, show that $T^2 \in \mathcal{K}(X)$. Is the converse true? Justify your answer. [12]

5. (a) Let $C_b(\mathbb{R})$ be the space of all bounded continuous functions f on $\mathbb{R}$ with the norm $\|f\| = \sup_{t\in\mathbb{R}} |f(t)|$. Define an operator T on $C_b(\mathbb{R})$ by

$$Tf(t) = f(t+c), \quad \forall t \in \mathbb{R},$$

where $c \in \mathbb{R}$ is a constant. Prove that $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

OR

(b) Let $K \subseteq \mathbb{C}$ be an arbitrary nonempty compact set. Construct an operator $T \in \mathcal{B}(l^2)$ such that $\sigma(T) = K$.

[15]

6. (a) State and prove Open Mapping Theorem.

OR

(b) Let X be a Banach space. Prove that X is infinite dimensional if and only if any totally bounded set in X is nowhere dense.

[15]

7. (a) State and prove Banach-Alaoglu Theorem.

(b) Show that every Banach space is isometrically isomorphic to a closed linear subspace of the space $C(X)$ of continuous functions on a compact Hausdorff space X.

[10+10]

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